Relativistic and non-relativistic equations of motion

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Abstract. It is shown that any second order dynamic equation on a configuration space X of non-relativistic time-dependent mechanics can be seen as a geodesic equation with respect to some (non-linear) connection on the tangent bundle $TX \to X$ of relativistic velocities. Using this fact, the relationship between relativistic and non-relativistic equations of motion is studied.

1 Introduction

In physical applications, one usually thinks of non-relativistic mechanics as being an approximation of small velocities of a relativistic theory. At the same time, the velocities in mathematical formalism of non-relativistic mechanics are not bounded. It has long been recognized that the relation between the mathematical schemes of relativistic and non-relativistic mechanics is not trivial. Our goal is the following.

Let X be a 4-dimensional world manifold of a relativistic theory, coordinated by (x^λ). Then the tangent bundle TX of X plays the role of a space of its 4-velocities (see Section 4). A relativistic equation of motion is said to be a geodesic equation

$$\ddot{x}^\mu = \dot{x}^\lambda \partial_\lambda \dot{x}^\mu = K^\mu_\lambda(x^\nu, \dot{x}^\nu)\dot{x}^\lambda \tag{1}$$

with respect to a (non-linear) connection

$$K = dx^\lambda \otimes (\partial_\lambda + K^\mu_\lambda \dot{\partial}_\mu) \tag{2}$$

on $TX \to X$. By $\dot{x}^\mu(x)$ in (1) is meant a geodesic vector field (which exists at least on a geodesic curve), while $\dot{x}^\lambda \partial_\lambda$ is the formal operator of differentiation. Throughout, we use the notation $\partial/\partial x^\lambda = \partial_\lambda$, $\partial/\partial \dot{x}^\lambda = \dot{\partial}_\lambda$.

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It is supposed additionally that there is a pseudo-Riemannian metric g of signature $(+,---)$ in TX such that a geodesic vector field does not leave the subbundle of relativistic hyperboloids

$$W_g = \{\dot{x}^\lambda \in TX \mid g_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu = 1\} \tag{3}$$

in TX. It suffices to require that the condition

$$(\partial_\lambda g_{\mu\nu}\dot{x}^\mu + 2g_{\mu\nu}K^\mu_\lambda)\dot{x}^\lambda\dot{x}^\nu = 0. \tag{4}$$

holds for all tangent vectors which belong to W_g (3). Obviously, the Levi–Civita connection $\{_\lambda{}^\mu{}_\nu\}$ of the metric g fulfills the condition (4). Any connection K on $TX \to X$ can be written as

$$K^\mu_\lambda = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\nu + \sigma^\mu_\lambda(x^\lambda, \dot{x}^\lambda),$$

where the soldering form

$$\sigma = \sigma^\mu_\lambda dx^\lambda \otimes \dot{\partial}_\lambda$$

plays the role of an external force. Then the condition (4) takes the form

$$g_{\mu\nu}\sigma^\mu_\lambda\dot{x}^\lambda\dot{x}^\nu = 0. \tag{5}$$

Let now a world manifold X admit a projection $X \to \mathbf{R}$, where $\mathbf{R}$ is a time axis. One can think of the bundle $X \to \mathbf{R}$ as being a configuration space of a non-relativistic mechanical system. It is provided with the adapted bundle coordinates (x^0, x^i), where the transition functions of the temporal one are $x'^0 = x^0$+const. The velocity phase space of a non-relativistic mechanics is the first order jet manifold J^1X of $X \to \mathbf{R}$, coordinated by (x^λ, x^i_0) [1, 4, 7, 9, 12]. There is the canonical imbedding of J^1X onto the affine subbundle of the tangent bundle TX, given by the coordinate condition

$$\dot{x}^0 = 1, \qquad \dot{x}^i = x^i_0 \tag{6}$$

(see (18) below). Then one can think of (6) as the 4-velocities of a non-relativistic system. The relation (6) differs from the relation (44) between 4- and 3-velocities of a relativistic system. In particular, the temporal component $\dot{x}^0$ of 4-velocities of a non-relativistic system equals 1 (relative to the universal unit system). It follows that the 4-velocities of relativistic and non-relativistic systems occupy different subbundles of the tangent bundle TX. We show the following.

Proposition 1. Let J^2X be the second order jet manifold of $X \to \mathbf{R}$, coordinated by $(x^\lambda, x^i_0, x^i_{00})$. Each second order dynamic equation

$$x^i_{00} = \xi^i(x^0, x^j, x^j_0) \tag{7}$$

of non-relativistic mechanics on $X \to \mathbf{R}$ is equivalent to the geodesic equation

$$\begin{aligned} &\dot{x}^\lambda \partial_\lambda \dot{x}^0 = 0, \qquad \dot{x}^0 = 1, \\ &\dot{x}^\lambda \partial_\lambda \dot{x}^i = \overline{K}^i_0 \dot{x}^0 + \overline{K}^i_j \dot{x}^j \end{aligned} \tag{8}$$

with respect to a connection $\overline{K}$ on $TX \to X$ which fulfills the conditions

$$\overline{K}^0_\lambda = 0, \qquad \xi^i = \overline{K}^i_0 + x^j_0 \overline{K}^i_j \mid_{\dot{x}^0=1, \dot{x}^i = x^i_0} . \tag{9}$$

Note that, written relative to bundle coordinates (x^0, x^i) adapted to a given fibration $X \to \mathbf{R}$, the connection $\overline{K}$ (9) and the geodesic equation (8) are well defined with respect to any coordinates on X. It should be also emphasized that the connection $\overline{K}$ (9) is not determined uniquely. We will return repeatedly to this ambiguity. This ambiguity is overcome if the relativistic transformation law of ξ is known.

Thus, we observe that both relativistic and non-relativistic equations of motion can be seen as the geodesic equations on the same tangent bundle TX. The difference between them lies in the fact that their solutions live in the different subbundles (3) and (6) of TX. At the same time, relativistic equations, expressed into the 3-velocities $\dot{x}^i/\dot{x}^0$ of a relativistic system, tend exactly to the non-relativistic equations on the subbundle (6) when $\dot{x}^0 \to 1$, $g_{00} \to 1$, i.e., where non-relativistic mechanics and the non-relativistic approximation of a relativistic theory coincide only.

2 Main relations

There is the following relationship between relativistic and non-relativistic equations of motion.

By a reference frame in non-relativistic mechanics is meant an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ such that the transition functions of the spatial coordinates x^i are independent of the temporal one x^0. In Section 3, we will give an equivalent definition of a reference frame as a connection Γ on $X \to \mathbf{R}$.

Given a reference frame (x^0, x^i), any connection $K(x^\lambda, \dot{x}^\lambda)$ (2) on the tangent bundle $TX \to X$ defines the connection $\overline{K}$ on $TX \to X$ with the components

$$\overline{K}^0_\lambda = 0, \qquad \overline{K}^i_\lambda = K^i_\lambda. \tag{10}$$

It follows that, given a fibration $X \to \mathbf{R}$, every relativistic equation of motion (1) yields the geodesic equation (8) and, consequently, has the counterpart

$$x^i_{00} = K^i_0(x^\lambda, 1, x^k_0) + x^j_0 K^i_j(x^\lambda, 1, x^k_0) \tag{11}$$

(7) in non-relativistic mechanics. Note that, written with respect to a reference frame (x^0, x^i), the connection $\overline{K}$ (9) and the corresponding geodesic equation (8) are well defined relative to any coordinates on X, while the dynamic equation (2) is done relative to arbitrary coordinates on X, compatible with the fibration $X \to \mathbf{R}$. The key point is that, for another reference frame (x^0, x'^i) with time-dependent transition functions $x^i \to x'^i$, the same connection K (2) on TX sets another connection $\overline{K}'$ on $TX \to X$ with the components

$$K'^0_\lambda = 0, \qquad K'^i_\lambda = \left(\frac{\partial x'^i}{\partial x^j} K^j_\mu + \frac{\partial x'^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda} + \frac{\partial x'^i}{\partial x^0} K^0_\mu \frac{\partial x^\mu}{\partial x'^\lambda}$$

with respect to the reference frame (x^0, x'^i). It is easy to see that the connection $\overline{K}$ (10) has the components

$$K'^0_\lambda = 0, \qquad K'^i_\lambda = \left(\frac{\partial x'^i}{\partial x^j} K^j_\mu + \frac{\partial x'^i}{\partial x^\mu}\right)\frac{\partial x^\mu}{\partial x'^\lambda},$$

relative to same reference frame. This illustrates the obvious fact that a non-relativistic approximation is not relativistic invariant (see, e.g. [5]).

The converse procedure is more intricate. At first, a non-relativistic dynamic equation (7) is brought into the geodesic equation (8) with respect to the connection $\overline{K}$ (9). A solution is not unique in general. Then, one should find a pair (g, K) of a pseudo-Riemannian metric g and a connection K on $TX \to X$ such $K^i_\lambda = \overline{K}^i_\lambda$ and the condition (4) is fulfilled.

From the physical viewpoint, the most interesting dynamic equations are the quadratic ones (sprays), i.e.,

$$\xi^i = a^i_{jk}(x^\mu) x^j_0 x^k_0 + b^i_j(x^\mu) x^j_0 + f^i(x^\mu). \tag{12}$$

This property is global due to the transformation law (21). Then one can use the following two facts.

Proposition 2. Any quadratic dynamic equation (12) is equivalent to the geodesic equation (8) for the symmetric linear connection

$$\overline{K} = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu(x^\alpha)\dot{x}^\nu \dot{\partial}_\mu)$$

on $TX \to X$, given by the components

$$K_\lambda{}^0{}_\nu = 0, \quad K_0{}^i{}_0 = f^i, \quad K_0{}^i{}_j = \frac{1}{2}b^i_j, \quad K_j{}^i{}_k = a^i_{jk}. \tag{13}$$

It follows that every non-relativistic quadratic dynamic equation

$$x^i_{00} = a^i_{jk}(x^\mu)x^j_0x^k_0 + b^i_j(x^\mu)x^j_0 + f^i(x^\mu) \tag{14}$$

(12) gives rise to the geodesic equation

$$\begin{aligned} &\dot x^\lambda\partial_\lambda \dot x^0 = 0, \qquad \dot x^0 = 1, \\ &\dot x^\lambda\partial_\lambda \dot x^i = a^i_{jk}(x^\mu)\dot x^i\dot x^j + b^i_j(x^\mu)\dot x^j\dot x^0 + f^i(x^\mu)\dot x^0\dot x^0 \end{aligned} \tag{15}$$

on a world manifold X.

Proposition 3. Every affine vertical vector field

$$\sigma = (b^i_j(x^\mu)x^j_0 + f^i(x^\mu))\partial^0_i$$

on the affine jet bundle $J^1X \to X$ is extended to the soldering form

$$\sigma = (f^i dx^0 + b^i_k dx^k) \otimes \dot\partial_i$$

on the tangent bundle $TX \to X$.

It follows that, in particular, if there is no topological obstruction and the Minkowski metric η on TX exists, a non-relativistic dynamic equation

$$x^i_{00} = b^i_j(x^\mu)x^j_0 + f^i(x^\mu) \tag{16}$$

gives rise to the geodesic equation

$$\begin{aligned} &\dot x^\lambda\partial_\lambda \dot x^0 = 0, \qquad \dot x^0 = 1, \\ &\dot x^\lambda\partial_\lambda \dot x^i = b^i_j(x^\mu)\dot x^j + f^i(x^\mu)\dot x^0. \end{aligned} \tag{17}$$

We meet the above-mentioned ambiguity. The non-relativistic dynamic equation (16) can be represented as both the geodesic equation (17) and the one (15), where $a = 0$. The first is the case for external forces, e.g., an electromagnetic theory, while the latter is that for a gravitation theory. One can also use the following assertion, instead of Proposition 3.

Proposition 4. Any non-relativistic quadratic dynamic equation (12), being equivalent to the geodesic equation with respect to the linear connection $\overline{K}$ (13), is also equivalent to the one with respect an affine connection K' on $TX \to X$ which differs from $\overline{K}$ (13) in a soldering form σ on $TX \to X$ with the components

$$\sigma^0_\lambda = 0, \qquad \sigma^i_k = h^i_k - \frac{1}{2}h^i_k \dot{x}^0, \qquad \sigma^i_0 = -\frac{1}{2}h^i_k \dot{x}^k - h^i_0 \dot{x}^0 + h^i_0,$$

where h^i_λ are local functions on X.

In Sections 3–4, we give a brief exposition of geometry of non-relativistic and relativistic mechanics and prove Propositions 1–3 (see [1, 7, 12] for details). Section 5 is devoted to several examples. In particular, we show that there is a coordinate system where the Lagrange equation for a non-degenerate quadratic Lagrangian in non-relativistic mechanics coincides with the non-relativistic approximation of the geodesic motion in the presence of some pseudo-Riemannian or Riemannian metric, whose spatial part is a mass tensor.

3 Geometry of non-relativistic mechanics

Let a fibre bundle $X \to \mathbf{R}$, coordinated by (x^0, x^i), be a configuration space of non-relativistic mechanics. Its base $\mathbf{R}$ is equipped with the standard vector field ∂_0 and the standard 1-form dx^0. The velocity phase space of non-relativistic mechanics is the first order jet manifold J^1X of sections c of $X \to \mathbf{R}$, which is provided with the adapted coordinates (x^0, x^i, x^i_0). Recall that J^1X comprises the equivalence classes $j^1_{x^0}c$ of $X \to \mathbf{R}$ which are identified by their values $c^i(x^0)$ and the values of their derivatives $\partial_0 c^i(x^0)$ at points $x^0 \in \mathbf{R}$, i.e.,

$$x^i_0(j^1_t c) = \partial_0 c^i(t).$$

There is the canonical imbedding

$$\lambda : J^1X \hookrightarrow TX, \qquad \lambda = \partial_0 + x^i_0 \partial_i, \tag{18}$$

over X. From now on, we will identify J^1X with its image in TX. It is an affine bundle modelled over the vertical tangent bundle VX of $X \to \mathbf{R}$.

In particular, every connection on a bundle $X \to \mathbf{R}$ is given by the nowhere vanishing vector field

$$\Gamma : X \to J^1X \subset TX, \qquad \Gamma = \partial_0 + \Gamma^i \partial_i, \tag{19}$$

on X. It can be treated as a reference frame in non-relativistic mechanics. Every connection Γ^i (19) defines an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ and the associated coordinates (x^0, x^i) on X such that the transition functions $x^i \to x'^i$ are independent of x^0, and *vice versa* [1]. We find $\Gamma^i = 0$ with respect to these coordinates. In particular, there is one-to-one correspondence between the complete connection Γ (19) and the trivializations $X \cong \mathbf{R} \times M$ of the configuration bundle X. Recall that different trivializations of X differ from each other in projections of X to its typical fibre M, while the fibration $X \to \mathbf{R}$ is one for all.

By a non-relativistic second order dynamic equation on a configuration bundle $X \to \mathbf{R}$ is meant the geodesic equation

$$x^i_{00} = \xi^i(x^\mu, x^j_0)$$

for a holonomic connection

$$\xi = \partial_0 + x^i_0 \partial_i + \xi^i(x^\mu, x^i_0)\partial^0_i \tag{20}$$

on the jet bundle $J^1X \to \mathbf{R}$. This connection takes its values into the second order jet manifold $J^2X \subset J^1J^1X$. It has the transformation law

$$\xi'^i = (\xi^j \partial_j + x^j_0 x^k_0 \partial_j \partial_k + x^j_0 \partial_j \partial_0 + \partial_0)x'^i. \tag{21}$$

Let us consider the relationship between the holonomic connections ξ (20) on the jet bundle $J^1X \to \mathbf{R}$ and the connections

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial^0_i) \tag{22}$$

on the affine jet bundle $J^1X \to X$. The connections γ have the transformation law

$$\gamma'^i_\lambda = (\partial_j x'^i \gamma^j_\mu + \partial_\mu x'^i_0)\frac{\partial x^\mu}{\partial x'^\lambda}. \tag{23}$$

Proposition 5. [1, 7]. Any connection γ (22) on the affine jet bundle $J^1X \to X$ defines the holonomic connection

$$\xi = \partial_0 + x^i_0 \partial_i + (\gamma^i_0 + x^j_0 \gamma^i_j)\partial^0_i \tag{24}$$

on the jet bundle $J^1X \to \mathbf{R}$.

It follows that every connection γ (22) on the affine jet bundle $J^1X \to X$ yields the dynamic equation

$$x^i_{00} = \gamma^i_0 + x^j_0 \gamma^i_j \tag{25}$$

on the configuration space X. Of course, different dynamic connections may lead to the same dynamic equation (25).

Proposition 6. [1, 7]. Any holonomic connection ξ (20) on the jet bundle $J^1X \to \mathbf{R}$ defines a connection

$$\gamma = dx^0 \otimes [\partial_0 + (\xi^i - \frac{1}{2}x^j_t\partial^t_j\xi^i)\partial^0_i] + dx^j \otimes [\partial_j + \frac{1}{2}\partial^0_j\xi^i\partial^0_i] \tag{26}$$

on the affine jet bundle $J^1X \to X$.

The connection γ (26), associated with a dynamic equation, possesses the property

$$\gamma^k_i = \partial^0_i\gamma^k_0 + x^j_0\partial^0_i\gamma^k_j \tag{27}$$

which implies $\partial^0_j\gamma^k_i = \partial^0_i\gamma^k_j$. Any connection γ, obeying the condition (27), is said to be symmetric.

Let γ be a connection (22) and ξ_γ the corresponding dynamic equation (24). Then the connection (26), associated with ξ_γ, takes the form

$$\gamma_{\xi_\gamma}{}^k_i = \frac{1}{2}(\gamma^k_i + \partial^0_i\gamma^k_0 + x^j_0\partial^0_i\gamma^k_j), \qquad \gamma_{\xi_\gamma}{}^k_0 = \xi^k - x^i_0\gamma_{\xi_\gamma}{}^k_i.$$

It is readily observed that $\gamma = \gamma_{\xi_\gamma}$ if and only if γ is symmetric.

Since the jet bundle $J^1X \to X$ is affine, it admits an affine connection

$$\gamma = dx^\lambda \otimes [\partial_\lambda + (\gamma^i_{\lambda 0}(x^\nu) + \gamma^i_{\lambda j}(x^\nu)x^j_0)\partial^0_i].$$

This connection is symmetric if and only if $\gamma^i_{\lambda\mu} = \gamma^i_{\mu\lambda}$. An affine connection γ generates a quadratic dynamic equation, and *vice versa*.

Now let us prove Proposition 1. We start from the relation between the connections γ on the affine jet bundle $J^1X \to X$ and the connections K (2) on the tangent bundle $TX \to X$ of the configuration space X. Let us consider the diagram

$$\begin{array}{ccc} J^1_XJ^1X & \xrightarrow{J^1\lambda} & J^1_XTX \\ \gamma \uparrow & & \uparrow K \\ J^1X & \xrightarrow{\lambda} & TX \end{array} \tag{28}$$

where J^1_XTX is the first order jet manifold of the tangent bundle $TX \to X$, coordinated by $(x^\lambda, \dot{x}^\lambda, \dot{x}^\lambda_\mu)$. The jet prolongation over X of the canonical imbedding λ (18) reads

$$J^1\lambda : (x^\lambda, x^i_0, x^i_{\mu 0}) \mapsto (x^\lambda, \dot{x}^0 = 1, \dot{x}^i = x^i_0, \dot{x}^0_\mu = 0, \dot{x}^i_\mu = x^i_{\mu 0}).$$

We have

$$J^1\lambda\circ\gamma : (x^\lambda, x^i_0) \mapsto (x^\lambda, \dot x^0 = 1, \dot x^i = x^i_0, \dot x^0_\mu = 0, \dot x^i_\mu = \gamma^i_\mu),$$
$$K\circ\lambda : (x^\lambda, x^i_0) \mapsto (x^\lambda, \dot x^0 = 1, \dot x^i = x^i_0, \dot x^0_\mu = K^0_\mu, \dot x^i_\mu = K^i_\mu).$$

It follows that the diagram (28) can be commutative only if the components K^0_μ of the connection K on $TX \to X$ vanish. Since the coordinate transition functions $x^0 \to x'^0$ are independent of x^i, a connection K with the components $K^0_\mu = 0$ can exist on the tangent bundle $TX \to X$. In particular, let (x^0, x^i) be a reference frame. Given an arbitrary connection K (2) on $TX \to X$, one can put $K^0_\mu = 0$ in order to obtain a desired connection

$$\overline{K} = dx^\lambda \otimes (\partial_\lambda + K^i_\lambda \dot\partial_i), \tag{29}$$

obeying the transformation law

$$K'^i_\lambda = (\partial_j x'^i K^j_\mu + \partial_\mu \dot x'^i)\frac{\partial x^\mu}{\partial x'^\lambda}. \tag{30}$$

Now the diagram (28) becomes commutative if the connections γ and K fulfill the relation

$$\gamma^i_\mu = K^i_\mu \circ \lambda = K^i_\mu(x^\lambda, \dot x^0 = 1, \dot x^i = x^i_0). \tag{31}$$

It is easily seen that this relation holds globally because the substitution of $\dot x^i = x^i_0$ into (30) restates the transformation law (23). In accordance with the relation (31), a desired connection $\overline{K}$ is an extension of the local section $J^1\lambda\circ\gamma$ of the affine bundle $J^1_X TX \to TX$ over the closed submanifold $J^1X \subset TX$ to a global section. Such an extension always exists, but it is not unique. Let us consider the geodesic equation (8) on TX with respect to the connection $\overline{K}$. Its solution is a geodesic curve $c(t)$ also satisfying the dynamic equation (7), and *vice versa*.

Remark. We can also consider the injection $J^2X \to TTX$, given by the coordinate relations

$$(x^\lambda, x^i_0, x^i_{00}) \mapsto (x^\lambda, \dot x^0 = \overset{\circ}{\mathrm{x}}{}^0 = 1, \dot x^i = \overset{\circ}{\mathrm{x}}{}^i = x^i_0, \ddot x^0 = 0, \ddot x^i = x^i_{00}), \tag{32}$$

and show that the dynamic equation (7) is equivalent to the restriction to (32) of a second order equation on X. However, one must prove that this dynamic equation is a geodesic equation. Recall that any second order dynamic equation

$$\ddot x^\lambda = \Xi^\lambda(x^\mu, \dot x^\mu) \tag{33}$$

on X defines a connection

$$K^\lambda_\mu = \frac{1}{2}\dot\partial_\mu \Xi^\lambda \tag{34}$$

on the tangent bundle $TX \to X$ [8, 11]. The equation (34) is a geodesic equation with respect to the connection (34) if and only if it is a spray. It is readily observed that, if (33) is a geodesic equation (1) with respect to a non-linear connection K, the corresponding connection (34) does not coincide with K in general.

Corollary 7. In accordance with the relation (31), every dynamic equation on the configuration space X can be written in the form

$$x^i_{00} = K^i_0 \circ \lambda + x^j_0 K^i_j \circ \lambda, \tag{35}$$

where $\overline{K}$ is a connection (29) on the tangent bundle $TX \to X$. Conversely, each connection $\overline{K}$ of the type (29) and, consequently, any connection K (2) on the tangent bundle $TX \to X$ defines a connection γ on the affine jet bundle $J^1X \to X$ and the dynamic equation (35) on the configuration space X.

Proposition 2 follows at once from the following lemma.

Lemma 8. There is one-to-one correspondence between the affine connections γ on the affine jet bundle $J^1X \to X$ and the linear connections $\overline{K}$ (29) on the tangent bundle $TX \to X$. This correspondence is given by the relation (31) which takes the form

$$\begin{aligned}
&\gamma^i_\mu = \gamma^i_{\mu 0} + \gamma^i_{\mu j} x^j_0 = K_\mu{}^i{}_0(x^\lambda)\dot x^0 + K_\mu{}^i{}_j(x^\lambda)\dot x^j|_{\dot x^0=1,\dot x^i = x^i_0} = K_\mu{}^i{}_0(x^\lambda) + K_\mu{}^i{}_j(x^\lambda) x^j_0,\\
&\gamma^i_{\mu\lambda} = K_\mu{}^i{}_\lambda.
\end{aligned} \tag{36}$$

In particular, if an affine connection γ is symmetric, so is the corresponding linear connection K.

Corollary 9. Any linear connection K on the tangent bundle $TX \to X$ defines the quadratic dynamic equation

$$x^i_{00} = K_0{}^i{}_0 + (K_0{}^i{}_j + K_j{}^i{}_0)x^j_0 + K_j{}^i{}_k x^j_0, x^k_0,$$

written with respect to a reference frame (x^0, x^i).

We conclude this Section with the proof of Proposition 3. Let ξ and ξ' be holonomic connections on the jet bundle $J^1X \to \mathbf{R}$. It is readily observed that their difference is a vertical vector field

$$\sigma = (\xi^i - \xi'^i)\partial_i^0$$

which takes its values into the vertical tangent bundle $V_XJ^1X \subset VJ^1X$ of the affine jet bundle $J^1X \to X$. Similarly to Lemma 8, one can show that there is one-to-one correspondence between the V_XJ^1X-valued affine vector fields

$$\sigma = (b^i_k(x^\mu)x^k_0 + f^i(x^\mu))\partial^0_i$$

on J^1X and the V_XTX-valued linear vertical vector fields

$$\overline{\sigma} = (b^i_k(x^\mu)\dot{x}^k + f^i(x^\mu)\dot{x}^0)\dot{\partial}_i$$

on TX. This linear vertical field determines a desired soldering form.

4 Geometry of relativistic mechanics

Let us consider a mechanical system whose configuration space X has not a preferable fibration $X \to \mathbf{R}$. We come to relativistic mechanics on X whose velocity phase space is the jet manifold of 1-dimensional submanifolds of X; that generalizes the notion of jets of sections of a bundle [1, 10].

Let Z be an $(m+n)$-dimensional manifold. The 1-order jet manifold J^1_nZ of n-dimensional submanifolds of Z comprises the equivalence classes $[S]^1_z$ of n-dimensional imbedded submanifolds of Z which pass through $z \in Z$, and are tangent to each other at z. It is provided with a manifold structure as follows.

Let $Y \to N$ be an $(m+n)$-dimensional bundle over an n-dimensional base N and Φ an imbedding of Y into Z. Then there is the natural injection

$$\begin{aligned} &J^1\Phi : J^1Y \to J^1_nZ, \\ &j^1_xs \mapsto [S]^1_{\Phi(s(x))}, \qquad S = \mathrm{Im}\,(\Phi \circ s), \end{aligned} \tag{37}$$

where s are sections of $Y \to N$. This injection defines a chart on J^1_nZ. Such charts with differentiable transition functions cover the set J^1_nZ. Therefore, one can utilize the following coordinate atlas on the jet manifold J^1_nZ of submanifolds of Z. Let Z be endowed with a manifold atlas with coordinate charts

$$(z^A), \qquad A = 1, \ldots, n+m. \tag{38}$$

Though J^0_nZ, by definition, is diffeomorphic to Z, let us provide J^0_nZ with the atlas where every chart (z^A) on a domain $U \subset Z$ is replaced with the charts on the same domain U which correspond to the different partitions of the collection (z^A) in collections of n and m coordinates, denoted by

$$(q^\lambda, y^i), \qquad \lambda = 1, \ldots, n, \qquad i = 1, \ldots, m. \tag{39}$$

The transition functions between the coordinate charts (39) of J^0_nZ, associated with the coordinate chart (38) of Z, reduce simply to exchange between coordinates q^λ and y^i. Transition functions between arbitrary coordinate charts of the manifold J^0_nZ read

$$\widetilde{q}^\lambda = \widetilde{g}^\lambda(q^\mu, y^j), \quad \widetilde{y}^i = \widetilde{f}^i(q^\mu, y^j), \quad q^\alpha = g^\alpha(\widetilde{q}^\mu, \widetilde{y}^j), \quad y^i = f^i(\widetilde{q}^\mu, \widetilde{y}^j). \tag{40}$$

Given the coordinate atlas (39) of the manifold J^0_nZ, the jet manifold J^1_nZ of Z is endowed with the adapted coordinates $(q^\lambda, y^i, y^i_\lambda)$. Using the formal total derivatives $d_\lambda = \partial_\lambda + y^i_\lambda \partial^i$, one can write the transformation rules for these coordinates in the following form. Given the coordinate transformations (40), it is easy to find that

$$d_{\widetilde{q}^\lambda} = \left[d_{\widetilde{q}^\lambda} g^\alpha(\widetilde{q}^\lambda, \widetilde{y}^i) \right] d_{q^\alpha}. \tag{41}$$

Then we have

$$\widetilde{y}^i_\lambda = \left[\left(\frac{\partial}{\partial \widetilde{q}^\lambda} + \widetilde{y}^p_\lambda \frac{\partial}{\partial \widetilde{y}^p} \right) g^\alpha(\widetilde{q}^\lambda, \widetilde{y}^i) \right] \left(\frac{\partial}{\partial q^\alpha} + y^j_\alpha \frac{\partial}{\partial y^j} \right) \widetilde{f}^i(q^\mu, y^j). \tag{42}$$

When $n = 1$, the formalism of jets of submanifolds provides the adequate mathematical description of relativistic mechanics. In this case, the fibre coordinates y^i_0 on $J^1_1Z \to Z$, with the transition functions (42), are exactly the familiar coordinates on the projective space $\mathbf{RP}^m$.

Let X be a 4-dimensional world manifold equipped with an atlas of coordinates (x^0, x^i) (39) together with the transition functions (40) which take the form

$$x^0 \to \widetilde{x}^0(x^0, x^j), \qquad x^i \to \widetilde{x}^i(x^0, x^j). \tag{43}$$

The coordinates x^0 in different charts of X play the role of the temporal ones.

Let J^1_1X be the jet manifold of 1-dimensional submanifolds of X. It is provided with the adapted coordinates (x^0, x^i, x^i_0). Then one can think of x^i_0 as being the 3-velocities of relativistic mechanics. Their transition functions are obtained as follows.

Given the coordinate transformations (43), the total derivative (41) reads

$$d_{\widetilde{x}^0} = d_{\widetilde{x}^0}(x^0) d_{x^0} = \left(\frac{\partial x^0}{\partial \widetilde{x}^0} + \widetilde{x}^k_0 \frac{\partial x^0}{\partial \widetilde{x}^k} \right) d_{x^0}.$$

In accordance with the relation (42), we have

$$\widetilde{x}_0^i = d_{\widetilde{x}^0}(x^0) d_{x^0}(\widetilde{x}^i) = \left(\frac{\partial x^0}{\partial \widetilde{x}^0} + \widetilde{x}_0^k \frac{\partial x^0}{\partial \widetilde{x}^k}\right)\left(\frac{\partial \widetilde{x}^i}{\partial x^0} + x_0^j \frac{\partial \widetilde{x}^i}{\partial x^j}\right).$$

The solution of this equation is

$$\widetilde{x}_0^i = \left(\frac{\partial \widetilde{x}^i}{\partial x^0} + x_0^j \frac{\partial \widetilde{x}^i}{\partial x^j}\right) \Big/ \left(\frac{\partial \widetilde{x}^0}{\partial x^0} + x_0^k \frac{\partial \widetilde{x}^0}{\partial x^k}\right).$$

To obtain the relation between 3- and 4-velocities of a relativistic system, let us consider the morphism

$$\rho : TX \to J_1^1 X, \qquad x_0^i \circ \rho = \dot{x}^i / \dot{x}^0. \tag{44}$$

It is readily observed that the coordinate transformation laws of x_0^i and $\dot{x}^i/\dot{x}^0$ are the same. Therefore, one can think of the tangent bundle TX as being the space of the 4-velocities of relativistic mechanics.

The morphism (44) is a surjection. Let us assume that the tangent bundle TX is equipped with a pseudo-Riemannian metric g such that X is time oriented. The bundle of hyperboloids W_g (3) is the disjoint union of two connected imbedded subbundles of W^+ and W^- of TX. Then the restriction of the morphism (44) to each of these subbundles is an injection into $J_1^1 X$.

Let us consider the image of this injection in the fibre of $J_1^1 X$ over a point $x \in X$. There are coordinates (x^0, x^i) in a neighbourhood of x such that a pseudo-Riemannian metric $g(x)$ at x is brought into the Minkowski one η. In this coordinates, the hyperboloid $W_x \subset T_x X$ is

$$(\dot{x}^0)^2 - \sum_i (\dot{x}^i)^2 = 1.$$

This is the union of the subsets W_x^+, where $x^0 > 0$, and W_x^-, where $x^0 < 0$. The image $\rho(W_x^+)$ is given by the coordinate relation

$$\sum_i (x_0^i)^2 < 1.$$

This relation means that the 3-velocities of a relativistic system (X, g) are bounded in accordance with the relativity principle.

It should be emphasized the difference between relativistic and non-relativistic 3-velocities. If a world manifold X admits a fibration $X \to \mathbf{R}$, there is the canonical imbedding

$$i: J^1X \to J^1_1X \tag{45}$$

of the velocity phase space J^1X of a non-relativistic system on $X \to \mathbf{R}$ to that of a relativistic system (cf. (37)). Moreover, we have $i = \rho \circ \lambda$. However, the image of the morphism i (45) differs from that of the morphism (44) for any pseudo-Riemannian metric g on TX.

5 Examples

In order to compare relativistic and non-relativistic dynamics, one should consider pseudo-Riemannian metric on TX, compatible with the fibration $X \to \mathbf{R}$. Note that $\mathbf{R}$ is a time of non-relativistic mechanics. It is one for all non-relativistic observers. In the framework of a relativistic theory, this time can be seen as a cosmological time. Given a fibration $X \to \mathbf{R}$, a pseudo-Riemannian metric on the tangent bundle TX is said to be admissible if it is defined by a pair (g^R, Γ) of a Riemannian metric on X and a non-relativistic reference frame Γ (19), i.e.,

$$g = \frac{2\Gamma \otimes \Gamma}{\mid \Gamma \mid^2} - g^R, \tag{46}$$
$$\mid \Gamma \mid^2 = g^R_{\mu\nu}\Gamma^\mu\Gamma^\nu = g_{\mu\nu}\Gamma^\mu\Gamma^\nu,$$

in accordance with the well-known theorem [2]. The vector field Γ is a time-like vector relative to the pseudo-Riemannian metric g (46), but not with respect to other admissible pseudo-Riemannian metrics in general.

Given a coordinate systems (x^0, x^i), compatible with the fibration $X \to \mathbf{R}$, let us consider a non-degenerate quadratic Lagrangian

$$L = \frac{1}{2}m_{ij}(x^\mu)x^i_0x^j_0 + k_i(x^\mu)x^i_0 + f(x^\mu), \tag{47}$$

where m_{ij} is a Riemannian mass tensor. Similarly to Lemma 8, one can show that any quadratic polynomial in $J^1X \subset TX$ is extended to a bilinear form in TX. Then the Lagrangian L (47) can be written as

$$L = -\frac{1}{2}g_{\alpha\mu}x^\alpha_0x^\mu_0, \qquad x^0_0 = 1, \tag{48}$$

where g is the metric

$$g_{00} = -2f, \qquad g_{0i} = -k_i, \qquad g_{ij} = -m_{ij}. \tag{49}$$

The corresponding Lagrange equation takes the form

$$x^i_{00} = -(m^{-1})^{ik}\{_{\lambda k\nu}\}x^\lambda_0 x^\nu_0, \qquad x^0_0 = 1, \tag{50}$$

where

$$\{_{\lambda\mu\nu}\} = -\frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\mu g_{\lambda\nu})$$

are the Christoffel symbols of the metric (49). Let us assume that this metric is non-degenerate. By virtue of Proposition 2, the dynamic equation (50) can be brought into the geodesic equation (15) on TX which reads

$$\begin{aligned} &\dot x^\lambda \partial_\lambda \dot x^0 = 0, \qquad \dot x^0 = 1, \\ &\dot x^\lambda \partial_\lambda \dot x^i = \{_\lambda{}^i{}_\nu\}\dot x^\lambda \dot x^\nu - g^{i0}\{_{\lambda 0\nu}\}\dot x^\lambda \dot x^\nu. \end{aligned} \tag{51}$$

Let us now bring the Lagrangian (47) into the form

$$L = \frac{1}{2}m_{ij}(x^\mu)(x^i_0 - \Gamma^i)(x^j_0 - \Gamma^j) + f'(x^\mu), \tag{52}$$

where Γ is a Lagrangian connection on $X \to \mathbf{R}$. This connection Γ defines an atlas of local constant trivializations of the bundle $X \to \mathbf{R}$ and the corresponding coordinates $(x^0, \overline{x}^i)$ on X such that the transition functions $\overline{x}^i \to \overline{x}'^i$ are independent of x^0, and $\Gamma^i = 0$ with respect to $(x^0, \overline{x}^i)$ (see Section 3). In this coordinates, the Lagrangian L (52) reads

$$L = \frac{1}{2}\overline{m}_{ij}\overline{x}^i_0 \overline{x}^j_0 + f'(x^\mu).$$

One can think of its first term as the kinetic energy of a non-relativistic system with the mass tensor $\overline{m}_{ij}$ relative to the reference frame Γ, while $(-f')$ is a potential. Let us assume that f' is a nowhere vanishing function on X. Then the Lagrange equation (50) takes the form

$$\overline{x}^i_{00} = \{_\lambda{}^i{}_\nu\}\overline{x}^\lambda_0 \overline{x}^\nu_0, \qquad \overline{x}^0_0 = 1,$$

where $\{_\lambda{}^i{}_\nu\}$ are the Christoffel symbols of the metric (49) whose components with respect to the coordinates $(x^0, \overline{x}^i)$ read

$$g_{ij} = -\overline{m}_{ij}, \qquad g_{0i} = 0, \qquad g_{00} = -2f'. \tag{53}$$

This metric is Riemannian if $f' > 0$ and pseudo-Riemannian if $f' < 0$. Then the spatial part of the corresponding geodesic equation

$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^0 = 0, \qquad \dot{\overline{x}}^0 = 1,$$
$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^i = \{_\lambda{}^i{}_\nu\} \dot{\overline{x}}^\lambda \dot{\overline{x}}^\nu$$

is exactly the spatial part of the geodesic equation with respect to the Levi–Civita connection of the metric (53) on TX. It follows that, as was declared above, the non-relativistic dynamic equation (53) describes the non-relativistic approximation of the geodesic motion in the Riemannian or pseudo-Riemannian space with the metric (53). Note that the spatial part of this metric is the mass tensor which may be treated as a variable [6].

Conversely, let us consider a geodesic motion

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = \{_\lambda{}^\mu{}_\nu\} \dot{x}^\lambda \dot{x}^\nu \tag{54}$$

in the presence of a pseudo-Riemannian metric g on a world manifold X. Let $(x^0, \overline{x}^i)$ be local hyperbolic coordinates such that $g_{00} = 1$, $g_{0i} = 0$. This coordinates are a non-relativistic frame for a local fibration $X \to \mathbf{R}$. Then the equation (54) has the non-relativistic limit

$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^0 = 0, \qquad \dot{\overline{x}}^0 = 1,$$
$$\dot{\overline{x}}^\lambda \partial_\lambda \dot{\overline{x}}^i = \{_\lambda{}^i{}_\nu\} \dot{\overline{x}}^\lambda \dot{\overline{x}}^\nu \tag{55}$$

which is the Lagrange equation for the Lagrangian

$$L = \frac{1}{2} \overline{m}_{ij} \overline{x}^i_0 \overline{x}^j_0,$$

describing a free non-relativistic mechanical system with the mass tensor $\overline{m}_{ij} = -g_{ij}$. Relative to another frame $(x^0, x^i(x^0, \overline{x}^j))$ associated with the same local splitting $X \to \mathbf{R}$, the non-relativistic limit of the equation (54) keeps the form (55), whereas the non-relativistic equation (55) is brought into the Lagrange equation (51) for the Lagrangian

$$L = \frac{1}{2} m_{ij}(x^\mu)(x^i_0 - \Gamma^i)(x^j_0 - \Gamma^j). \tag{56}$$

This Lagrangian describes a mechanical system in the presence of the inertial force associated with the reference frame Γ. The difference between (51) and (55) shows that a gravitational force can not model an inertial force in general; that depends on

both a frame and a system. For example, if the mass tensor in the Lagrangian L (56) is independent of time, the corresponding Lagrange equation is a spatial part of the geodesic equation in a pseudo-Riemannian space.

In view of the ambiguity that we have mentioned, the relativization (48) of an arbitrary non-relativistic quadratic Lagrangian (47) may lead to a confusion. In particular, it can be applied to a gravitational Lagrangian (52) where

$$f' = -\frac{1}{2} + \phi,$$

and ϕ is a gravitational potential. An arbitrary quadratic dynamic equation can be written in the form

$$x^i_{00} = -(m^{-1})^{ik}\{_{\lambda k\mu}\}x^\lambda_0 x^\mu_0 + b^i_\mu(x^\nu)x^\mu_0, \qquad x^0_0 = 1,$$

where $\{_{\lambda k\mu}\}$ are the Christoffel symbols of some admissible pseudo-Riemannian metric g, whose spatial part is the mass tensor $(-m_{ik})$, while

$$b^i_k(x^\mu)x^k_0 + b^i_0(x^\mu) \tag{57}$$

is an external force. With respect to the coordinates where $g_{0i} = 0$, one may construct the relativistic equation

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = \{_\lambda{}^\mu{}_\nu\}\dot{x}^\lambda \dot{x}^\nu + \sigma^\mu_\lambda \dot{x}^\lambda, \tag{58}$$

where the soldering form σ must fulfill the condition (5). It takes place only if

$$g_{ik}b^i_j + g_{ij}b^i_k = 0,$$

i.e., the external force (57) is the Lorentz-type force plus some potential one. Then, we have

$$\sigma^0_0 = 0, \qquad \sigma^0_k = -g^{00}g_{kj}b^j_0, \qquad \sigma^j_k = b^j_k.$$

The "relativization" (58) exhausts almost all familiar examples. It means that a wide class of mechanical system can be represented as a geodesic motion with respect to some affine connection in the spirit of Cartan's idea. To complete our exposition, point out also another "relativization" procedure. Let a force $\xi^i(x^\mu)$ in the non-relativistic dynamic equation (7) be a spatial part of a 4-vector ξ^λ in the Minkowski space (X, η). Then one can write the relativistic equation

$$\dot{x}^\lambda \partial_\lambda \dot{x}^\mu = \xi^\lambda - \eta_{\alpha\beta}\xi^\beta \dot{x}^\alpha \dot{x}^\lambda. \tag{59}$$

This is the case, e.g., for a relativistic hydrodynamics that we meet usually in the literature on a gravitation theory. However, the non-relativistic limit $\dot{x}^0 = 1$ of (59) does not coincide with the initial non-relativistic equation. There are also other variants of relativistic hydrodynamic equations [3].

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